STREAM-PROFILE ANALYSIS AND STREAM-GRADIENT INDEX

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Abstract.—The generally regular three-dimensional geometry of drainage networks is the basis for a simple method of terrain analysis providing clues to bedrock conditions and other factors that determine topographic forms. On a reach of any stream, a gradient-index value can be obtained which allows meaningful comparisons of channel slope on streams of different sizes. The index is believed to reflect stream power or competence and is simply the product of the channel slope at a point and channel length measured along the longest stream above the point where the calculation is made. In an adjusted topography, changes in gradient-index values along a stream generally correspond to differences in bedrock or introduced load. In any landscape the gradient index of a stream is related to total relief and stream regimen. Thus, climate, tectonic events, and geomorphic history must be considered in using the gradient index. Gradient-index values can be obtained quickly by simple measurements on topographic maps, or they can be obtained by more sophisticated photogrammetric measurements that involve simple computer calculations from x, y, z coordinates.

Stream networks have regular geometric properties that can be quantitatively described. In an erosionally graded landscape (Hack, 1960, p. 89), the longitudinal profile of a stream is a property of stream geometry that can provide clues to underlying materials as well as insights into geologic processes and the geomorphic history of an area. This paper deals with a method of analyzing the longitudinal profile and develops a new unit of measurement, here called the stream-gradient index. This index relates the slope of a stream at a locality to its length at the locality and provides a basis for comparing stream reaches of different sizes.

During a study of stream profiles and streambed material in the Shenandoah Valley, Va. (Hack, 1957a, p. 87–90), it was found that a simple logarithmic graph of the stream profile provides the basis for a useful system of analysis. In this graph, the origin of the profile is at the drainage divide, which forms the source of the stream. The vertical coordinate is an arithmetic scale and represents the altitude or height above a datum. The horizontal coordinate is a logarithmic scale and represents the stream length or distance from the source.

Where the stream profile is a straight line on such a plot, the profile equation is

\[ H = C - k \log_6 L \]  

(1)

where \( H \) is altitude at a point on the profile and \( L \) is stream length (horizontal distance from the drainage divide to the same point on the principal stream measured along the channel). \( C \) and \( k \) are constants. The tangent to the profile or slope, \( S \), of the stream at a point is the derivative of equation 1,

\[ \frac{dH}{dL} = kL^{-1} \], or \[ S = \frac{k}{L} \].  

(2)

The slope equation can be put in the form \( SL = k \).

Most natural streams do not have a logarithmic profile such as the above throughout their length, but their profiles are made up of connected series of segments of various lengths, each logarithmic in form. The value of \( k \) (equivalent to the product \( SL \)) thus differs along the stream as a whole but is constant for any particular logarithmic segment. Because the value of \( k \) defines the steepness of the logarithmic profile for such a segment, it can be considered as an index of the relative steepness of the actual profile at a point. The product \( SL \) (equivalent to the constant \( k \) in equation 1) is here called the gradient index. As will be shown, this index is useful for streams as much as several hundred miles long, regardless of the overall form of the profile.

The gradient index is a significant quantity because it is crudely related to the power of a stream to transport material of a given size and to the characteristics of the channel that resist flow. This conclusion is supported by empirical evidence. It was found, for example, that streams in the Potomac basin on the average have the following relation between particle size on the bed, channel length, and channel slope:

\[ S = 25 \frac{M^{0.6}}{L} \].  

(3)

where \( S \) is slope, \( M \) is mean particle size on the bed (in millimeters) and \( L \) is length (in miles). Where \( M \) is constant in the equation above for a given reach of a stream, the product \( SL \) (gradient index) remains constant. An increase in particle size \( (M) \) is reflected in an increase in the product \( SL \) (see Hack, 1957a, p. 71). Several streams with nearly constant \( SL \) values over long reaches were found to have bed material that showed...
no systematic change in size within these reaches. The North Fork of the Shenandoah River (Hack and Young, 1959, p. 4) is an example. It has a narrow range of SL values and of particle sizes on the bed for a distance of about 100 miles (Hack and Young, 1959, p. 4). The Calf pasture River is another example and is discussed below. Thus, particle size clearly has a functional relation to gradient index.

The relation of the gradient index to the hydraulic geometry of stream systems is not understood. The fact that the product of slope and length has a relation to stream competence or power must be explainable in terms of the complex interrelations in the hydraulic geometry that characterize the conditions of dynamic equilibrium in streams. The theoretical aspects of channel equilibrium as related to the profile were discussed by Leopold, Wolman, and Miller (1964, p. 248–281). They stated (1964, table 7-7) that when power is equally distributed along a stream the product of slope and discharge is equal in all the reaches. Thus, if equal gradient indices are equatable with equal power, discharge must be proportional to length. Leopold, Wolman, and Miller's data show that, on the average, stream length is proportional to the 0.64 power of the drainage area. Also, bankfull discharge (which is thought to be the discharge that performs the significant work) is, on the average, proportional to the 0.75 power of the drainage area. Using these values, it is clear that bankful discharge is proportional to the 1.17 power of the channel length in the average stream. However, the data from which these figures are derived include streams that have varied flow resistance and varied power along their course and that on the average probably show a decrease in flow distance downstream. If the data were available, one might possibly be able to show that where flow energy remains the same in a downstream direction, length and bankfull discharge would be found to be nearly directly proportional.

**CALCULATION OF GRADIENT INDEX**

The gradient index can be measured on topographic maps, on aerial photographs using photogrammetric methods, or by ground surveys. The parameters measured are shown in figure 1. The quantity \( L \) is the stream length measured from the drainage divide at the source of the longest stream in the drainage basin above a locality on a reach. \( \Delta H \) is the difference in elevation between the ends of the reach, and \( \Delta L \) is the length of the reach. The reach must be long enough so that sharp local changes in slope such as those between riffles and pools are averaged out. This distance is, of course, much longer on large streams than on small ones.

The formula for the gradient index (SL) is \( SL = \frac{\Delta H \times L}{\Delta L} \).

Inasmuch as \( L/\Delta L \) is a dimensionless ratio, the measurements of horizontal distance can be in any unit of measure. The measurements are generally made on a map with a map measure, so it is convenient to use inches. It is not necessary to make the conversion to ground scale. As the contours on United States maps are generally in feet, the quantity \( \Delta H \) will be calculated in feet. The gradient index is the product of a ratio and a distance, so it is conveniently expressed as gradient-feet.

When measured in this way, the gradient index is equivalent to the product of the natural tangent to the channel slope at a locality and the distance in feet from the head of the stream to that locality. To find the value of the tangent expressed as a decimal, at any point, simply divide the index in gradient-feet by the distance in feet. The slope in feet per mile can be found by dividing the gradient index by the distance in miles. Thus, if the gradient index is 200 gradient-feet at 1 mile from the source, the slope will be 200 feet per mile; at 2 miles it will be 100 feet per mile. In the metric system, the gradient-index values would be different, as they would depend on measuring the difference in elevation at the ends of the reach in metric units rather than in feet.

Figure 2 is from an actual contour map and shows the measurements needed to find the gradient index at locality D in a reach extending from points A to B. \( \Delta H \) is 40 feet, the value of the contour interval. \( \Delta L \) is the distance from A to B, and \( L \) is the distance from C to D, including the bends in the stream. The distances can be measured on the map in any units desired, as long as the two measurements are in the same units.

As the validity of the gradient index depends on the general consistency of the relation between drainage area and channel length that exists in most natural stream systems, it is important that the distance \( L \) be measured along the principal (or longest) stream above the locality. The longest stream in the watershed must be followed continuously from its head to the terminating point at the locality. The operator must always proceed downstream from the head and never continue past a stream junction with a longer stream than the one being measured. A typical sequence of measurements is shown in figure 3.

Caution must be observed in calculating the SL value by the method above. The channel slope in the product SL is really a tangent to the profile at a particular point. In estimating the slope by measuring difference in height and length between the two ends of a reach, however, one is really measuring a
secant of a curve. If the reach is long, this secant will not be parallel to the tangent, and the error will increase as the length of the reach increases. The error is a function of the dimensionless ratio $L/\Delta L$ and becomes larger as this ratio becomes smaller. Significant error is probably not introduced until this ratio approaches 2.0 or even 1.0. At a stream length of 100 miles, for example, the use of a secant from length 50 miles to length 150 miles would introduce an error of only about 10 percent. On the other hand, one must be careful about obtaining average SL values near the head of a stream where the curvature is high. $\Delta L$ must be kept smaller than $L$.

The gradient index can also be found in terms of the profile equation 1, $H = C - k \log_{10} L$. The gradient index is equal to the constant, $k$, and can be found by the formula

$$k = \frac{H_1 - H_2}{\log_{10} L_2 - \log_{10} L_1},$$

where $H_1$ and $H_2$ are map elevations at each end of the reach measured, and $L_1$ and $L_2$ are distances from the source to each end of the reach.

**GRADIENT-INDEX CHANGES ALONG INDIVIDUAL STREAMS**

Individual profiles can be studied graphically by simple plotting on semilogarithmic graph paper or by converting the length measurements to logarithms and plotting on arithmetic scales. In practice, this is easily done by running down the map or mosaic of maps with a map measure and recording the contour elevations and their distances from the head of the stream. The data are then plotted. On such a logarithmic graph the gradient index is proportional to the slope of the plotted profile. Its numerical value can be calculated from the tabulated data at any point. Examples of several logarithmic profiles are shown and interpreted below.

**Calhoun River.**—The Calhoun River in Virginia, a headwater tributary of the James River, is, like the North Fork of the Shenandoah River (see page 422), an example of a stream whose logarithmic profile is close to a straight line (fig. 4). The average gradient index is 260 gradient-feet but it ranges from 210 to 300. The river valley is anticlinal, floored by Devonian shale, and surrounded by mountain ridges of
sandstone of Devonian and Mississippian age. Throughout the course, cobbles of sandstone are brought in by tributaries to make up the bed material, and the stream is bordered by low alluvial terraces of rounded sandstone cobbles. Shale crops out at only a few places in the stream and makes up only a very small proportion of the bed material, presumably because it breaks up into sizes too small to remain with sandstone cobbles in this part of the channel. Samples of the bed material at 10 localities along the stream range in size from 42 to 75 mm. The average size of the 10 samples is 61 mm.

The uniformity in size of the bed material suggests that the stream has uniform competence along its course because of the bedrock geology of the basin. Data on sizes of bed material are from Hack (1957a, table 8).

North River.—The North River in Virginia, the principal headwater of the South Fork, Shenandoah River, heads on Shenandoah Mountain and drains about 50 sq mi of country underlain largely by Devonian and Mississippian sandstone beds (fig. 4). It enters the limestone and shale lowland of the Shenandoah Valley about 18 miles from its source. At this point the bed material is more than 100 mm in median size. As the North River enters the Shenandoah Valley, its flood plain and terraces widen out to form a conspicuous lowland alluvial plain 2 miles wide which gradually narrows again downstream. At mile 36 the stream becomes entrenched in the limestone valley floor, and its flood plain is narrow. The size of the bed material systematically decreases from 110 mm to 42 mm (Hack, 1957a, table 8). As shown in figure 4, the profile is sharply curved on a logarithmic graph, and the values of the gradient index decrease from 1,000 to 250 as the gradient flattens. The bed material decreases in size with distance from the sandstone beds that are the source of the resistant material. Thus, the geology of the drainage basin controls the profile, as the competence and the slope are adjusted to handle the load.

Gillis Falls.—This stream, a headwater of the Patapsco River on the Maryland Piedmont, is an example of a stream with a profile of low concavity that shows an increase in the gradient index downstream (fig. 4). Size of bed material also increases downstream. This is because the watershed is underlain by soft, weathered phyllite laced by stringers and veins of resistant quartz. Though widespread in the watershed, the total volume of the quartz veins is very small in comparison with phyllite and weathered phyllite. The phyllite breaks up into small chips averaging less than 5 mm in size, whereas the quartz fragments average 80 to 100 mm. In the upstream reaches, vein quartz makes up only a small part of the bedload but forms a lag deposit on the bed and becomes more and more concentrated downstream where it constitutes the major part of the bedload. The fine-grained phyllite particles are
carried out of the channel in the lower reaches. The increase in size of the bed material from 7.0 mm to 80 mm is fairly systematic. The banks of the stream are loamy clay rather than sand, and the high depth-width ratio decreases from a minimum of 0.52 at 1.7 mile to 0.05 at 11.6 miles. The high depth-width ratio probably accounts in part for the low gradient of this stream in comparison with the others just described. As shown in figure 4, the logarithmic profile graph curves downward. This does not mean that the profile is in reality convex upward but rather that the natural profile is less concave upward than the others. Note that calculated values of the product $SL$ increase downstream as the competence increases.

**Cranberry River.**—Major differences in the materials along the course of a stream commonly cause abrupt changes in the profile, as exemplified by the Cranberry River, Ontonagon County, Mich. (fig. 5; Hack, 1965, fig. 12, and p. B22–B28). In its upper reaches this river flows through an area of till and lake deposits. At 11.5 miles it enters a buried bedrock high, and the profile abruptly changes character. The $SL$ or $k$ value changes from 240 to 350. At mile 16 the river again enters till, and again the profile abruptly changes.

**Lower Potomac River.**—The streams discussed on the preceding pages are small, and their profiles appear to be adjusted to the geology of the terrain drained by them. In contrast, the Potomac is a very large river, and its lower course has apparently been affected by changes in base level during the late geologic history of the river. A logarithmic profile of the river below the junction with the South Branch is shown in figure 6. The data for this profile are from a survey by the U.S. Engineer Office, but distances are measured from the head of the South Branch, the longest headwater, in West Virginia. Because this profile is of the lower reaches of a long river, the logarithmic profile does not show much distortion as compared with a natural profile. The amount of distortion can be judged by comparing the logarithmic and true horizontal distance scales.

In the upper part of the profile, above the Shenandoah River, $SL$ values are comparable with the values of smaller Appalachian streams adjusted to similar rocks. Within the Valley and Ridge province below the South Branch, the river meanders through a sandstone and shale lowland. The most resistant rocks having an appreciable outcrop area along this stretch are in the Hampshire Formation, thinly interbedded shale and sandstone. Very resistant rock formations such as the Tuscarora and Pocono Sandstones also occur in the section, but they do not crop out near the channel. $SL$ values average 490 gradient-feet, a figure similar to values for those streams in the Appalachian Valley that drain large areas of sandstone and shale and where most of the bed material consists of sandstone.

In the carbonate-rock area of the Appalachian Valley, the average $SL$ value in the Potomac River is 429, which is higher than might be expected in carbonate rocks. However, it drops to 210 near Williamsport, Md., where the river crosses the Martinsburg Shale and to 220 in the Cambrian limestone below Shepherdstown, W. Va. An anomalously high value in the intervening reach near Dam No. 4 between Williamsport and Shepherdstown corresponds to the outcrop of the Beckmantown Dolomite, which in many places contains abundant chert.

The bed material is not readily sampled in the Potomac River because of the deep water. A random sample of part of the streambed was obtained, however, at a riffle near Shepherdstown, at the site of a onetime ford. Eighty-one fragments were measured and classified. The median size was 180 mm. The sample had the following lithologic composition: sandstone, 57 percent; limestone, 23 percent; chert, 12 percent; and not identified, 8 percent. The nearest source of sandstone is in the Valley and Ridge province, 30 to 40 miles upstream. This composition suggests that most of the bed material consists of resistant Devonian and Silurian sandstones and that they are largely responsible for the unusually steep channel slopes of the Appalachian Valley part of the stream. The average $SL$ values obtained in other streams in carbonate rocks of the Shenandoah Valley are 180 in small streams and 245 in large streams. Large streams generally carry some resistant cobbles derived from headwater tributaries.

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Figure 6.—Profile of the Potomac River from its junction with the South Branch to Washington, D.C. Data from Somervell, 1929 (see footnote 1).

Gradient index in parentheses.

The Potomac profile steepens markedly at the water gap in the Blue Ridge. Here the river passes through the resistant quartzites and other clastic rocks of the Chilhowee Group, the metavolcanic of the Catoctin Formation, and other crystalline rocks. An SL value of 2,500 seems high compared with SL values found in smaller streams in the same rocks in other places. However, this high value may simply reflect adjustment of the slope to high flow resistance encountered at the crossings of tough rock formations. In the Triassic basin between the Monocacy River and Seneca Creek, SL values drop again to 250, a value comparable with small streams in predominantly shale areas.

At Seneca Creek, Md., the channel slope steepens abruptly and becomes a series of rapids and falls. In this reach the average SL is 2,900, far higher than in the smaller streams of the adjacent Piedmont where SL values average less than 300. Great Falls itself corresponds to an outcrop of resistant quartz-rich gneiss (U.S. Dept. Interior, 1970, p. 11–15), so there has been some adjustment by the river to the relative resistance of the rocks. However, the anomalously high SL values suggest that in this reach the river is also adjusted to a lower base level than are the reaches upstream. In this connection, marine deposits of Miocene age at Ward Creek in Washington, D.C., are at 370 feet above sea level (Darton, 1950, p. 6). Marine deposits of supposed Pliocene age at Good Hope Hill in eastern Washington, D.C., are at 280 feet (Hack, 1955). The Pleistocene channel of the Potomac River dropped 90 feet below sea level at Alexandria, Va., approximately 12 miles downstream from Chain Bridge (Hack, 1957b, p. 823) indicating a change in base level of more than 370 feet in the last million years. Thus, the lowest part of the Potomac River gorge is probably partly adjusted to low sea level of late Wisconsin time corresponding to the 10-foot channel at Alexandria.

Two conclusions can be drawn from the profile of the Potomac River: (1) That part of the profile above the Shenandoah River is adjusted to the geology of the basin and to the profiles of other streams in the upper basin. This adjustment may extend as far as Seneca. The reaches below Seneca, however, reflect accelerated erosion related to a drop in sea level in Pliocene and post-Pliocene time. (2) The general conformity of the gradient-index values above the Shenandoah River to those of tributary streams indicates that there is probably little, if any, tendency for SL values to change systematically because of geometric factors related simply to the size of a stream. This conclusion is important to the usefulness of the gradient index as an analytical tool. Still larger streams, however, such as the Mississippi, may have anomalously high SL values. The Mississippi River is so large that the overall geometry of its basin is probably determined by factors such as the configuration of the continent and the location of mountain ranges, rather than by the evolutionary development of drainage that normally proceeds with downwasting.
REGIONAL PLOTTING OF GRADIENT-INDEX VALUES ON MAPS

Indices of streams can be studied on a regional basis by plotting regularly spaced index values on maps; contour lines can then be drawn. Several such maps have been constructed.

The procedure is to run down with a map measure all the streams in the area studied, systematically calculating SL values at regular intervals. It is important in doing this that stream length always be measured along the longest channel in a given basin. That is, the measurements must be ordered as shown in figure 3. The spacing of measurement points is limited by the scale and detail of the topographic maps used in the calculations as well as by the actual spacing of the streams. In the Piedmont and Appalachian regions, for example, stream channels range on the average from 0.2 to 1.0 mile apart, depending on drainage density in a region, a factor generally related to the kind of rock. Measurement localities cannot be closer to the source of a stream than about ½ mile, because close to the divide the channel is not sculptured primarily by the kinetic energy of flowing water.

Figure 7 is an example of a gradient-index map. It is based on about 400 localities measured on four 7½-minute quadrangles with 20- to 40-foot contour intervals at 1:24,000 scale. The map covers the Lenoir 15-minute quadrangle, North Carolina. It was made as part of a geomorphic study of the Blue Ridge escarpment in the Grandfather Mountain area. The problem of the Blue Ridge escarpment is not discussed here, but features shown on this map are pointed out as examples of interpretations that can be made.

First of all, the base of the Blue Ridge escarpment in this area corresponds closely to the trace of the Brevard zone which extends from A on the left margin of the map to A′ at the top. The Piedmont province is south and east of that zone (A–A′) and is underlain by lower Paleozoic metamorphic rock covered by saprolite. Northwest of the Brevard zone, the relief increases sharply, and rock outcrops are more numerous. In this area the rocks are gneisses of the Grandfather Mountain window.

Note the anomalously high values at B north of the Johns River. These correspond to several lenses of extremely resistant quartz-rich gneiss (Bryant and Reed, 1970, p. 66). The anomalously high SL values led the writer to them. The lenses were poorly exposed, generally along the bottoms of deep valleys often difficult to reach because they were covered by slash from logging operations, but they were associated with steep places in the streams.

At C, a large tributary of the Johns River almost as long as the Johns River itself, the values are anomalously high. This area corresponds to the outcrop of the Brown Mountain Granite (Bryant and Reed, 1970). The granite is a particularly massive rock with widely spaced joints. The river cuts through it in a deep picturesque gorge with many cascades.

SL values drop sharply in the Piedmont southeast of A–A′. An interesting feature of this region is that the large streams have higher index values than their tributaries, though the index values of the tributaries increase as they approach the large streams. The probable explanation is that the larger streams are adjusted to transport resistant material derived from resistant rocks upstream. They, therefore, require steeper slopes for a given length. The smaller streams originate in saprolite on uplands of low relief and therefore are competent to transport only fine material. The situation is analogous to that described on page 424, with reference to Gillis Falls, Md. It is a common phenomenon in the Piedmont region where the rocks are complex mineralogically. They thus have several components of varying resistance. The more resistant components, such as fragments of quartz veins or aplite dikes, concentrate downstream. As these components wear more slowly than others, the size of the bed material tends to be
larger in the larger streams because the resistant components survive a greater distance of transport as large fragments and become relatively more abundant.

Maps similar to figure 7 can be constructed by photogrammetric measurements. By this method the spacing of points on streams is not limited by the contour interval but only by the spacing of the streams themselves and the resolution of the photogrammetric model. By use of photogrammetric equipment it is possible to plot and calculate gradient-index values automatically. A test on a model of an area in Montgomery County, Md., was made by using a Halcon stereoplotter, an optical projection instrument similar to the Keleb plotter. The plotter was equipped with a digital readout system that provided punched cards with the $x$, $y$, and $z$ coordinates of each measured point at model scale. A Haag-Streit coordinate plotter equipped with digital output in card form was used to measure points on the topographic map to be used as control for absolute orientation of the model. Data reduction and stream-gradient computations were made on an IBM 1130 computer. Graphic presentations of the results were plotted on a Cal-Comp plotter operated on line with the 1130 computer. Part of the graphic presentation is shown in figure 8.

**Gradient Index in Different Climatic and Geomorphic Regions**

The examples of the gradient index cited are all in the Eastern United States, a humid region in which the density of perennial streams is high. The writer has had little experience with the index in arid regions, but it is questionable whether it would have the same meaning as it does in humid regions.

Although the index defines the slope of the channel relative to the length in arid regions, it would not necessarily correlate directly with stream power and flow resistance because the hydraulic geometry of the channel is different. For example, channel width may increase downstream in arid regions at different rates from those of humid regions because of the sparse vegetation on the banks. Presumably this is a factor influencing slope. Another factor is the relation of discharge to drainage area and length. In arid regions average discharge, as well as flood discharge, increases downstream at a regular rate only in small drainage areas a few square miles in size. Small areas are affected by individual storm cells. Although few data are available, in some larger drainage basins average discharge may actually decrease downstream.
Interpretation of the gradient index must differ to some extent in different geomorphic provinces. In the Eastern United States, most of the area is drained by converging streams with an average length proportional to the 0.64 power of the drainage area, a relation probably common to most stream-eroded landscapes in which tributaries converge downstream (Hack, 1957a, p. 63; Leopold, Wolman, and Miller, 1964, p. 144). In the Basin and Range province, however, and in other parts of the Western United States, long reaches of streams cross pediment surfaces, alluvial fans, and bajadas in which the geometric relations are different. Though the gradient index may be useful for analysis of such regions, it cannot be used in the same way as in most areas of the Eastern United States because the rates of increase in discharge downstream and the relation of the profile to stream power will not be similar.

CONCLUSION

The gradient index is useful in terrain analysis because it permits comparisons to be made of streams of different sizes. It is a sort of rule of thumb index of stream power and flow resistance. Because it is so simple to calculate from a topographic map, one of its most valuable uses may be to discover anomalously steep or gently reaches of streams that are related to particularly resistant or soft rocks. The index may thus serve as an aid to geologic mapping, analogous to data on magnetic intensities.

Regional analysis of stream profiles can help in interpreting the geomorphology of large regions. Playfair (1802, p. 114) first enunciated the idea that streams sculpture the valleys in which they flow, that their slopes are adjusted to carry away the debris in their watersheds, and that the whole river network is an adjusted system. The gradient index, when used in the general context of these ideas, can provide insights into the causes of the diversity of the landscape, including the nature of the adjustments that streams have made as well as the nature of diastrophic forces opposed to them.

REFERENCES CITED


